JOURNAL OF COMBINATORIAL THEORY 5, 90–103 (1968)

Forests of Labeled Trees

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ABSTRACT

Forests of labeled trees, rooted or not, oriented or not, are enumerated by number of (labeled) points and by number of trees for unrestricted and various restricted heights.

1. INTRODUCTION

The relation of a forest to its trees is the same as the relation of a linear graph to its connected parts. Indeed, a tree is the simplest connected graph, and a forest is a linear graph, all of whose parts are trees. Hence, it is not surprising that the simplicity of E. N. Gilbert's formula* [2]

$$L(x, y) = \exp C(x, y) \tag{1}$$

relating the enumerators of linear graphs and connected linear graphs (not necessarily without slings or lines in parallel) by number of points and number of lines, each with all points distinctly labeled, is preserved in the formula

$$F(x, y) = \exp xT(y) \tag{2}$$

relating the enumerators of forests of labeled trees by number of trees and number of points and the labeled trees by number of points. More precisely

$$F(x, y) = \sum_{n=0}^{\infty} \sum_{k=1}^{n} F_{nk} x^k y^n / n!$$

$$T(y) = \sum_{n=1}^{\infty} T_n y^n / n!$$

* A similar formula for graphs without slings and lines in parallel had been given earlier by R. J. Riddell, Contributions to the Theory of Condensation (dissertation, University of Michigan, 1951). A rapid derivation appears in G. W. Ford and G. E. Uhlenbeck, Combinatorial Problems in the Theory of graphs, I, *Proc. Nat. Acad. Sci.* **42** (1956), 122–128.

with F_{nk} the number of forests with n labeled points and k trees and T_n the number of trees with n labeled points. Equation (2) is a simple transcription to present terminology of equation (1) of my paper [6]. Note that the trees in question are limited only by the labeling requirement, that is, by having all points distinct; they may be rooted or planted or free, they may be oriented or directed or neither, they may be limited as to height, diameter, or degree specification. All specifications of the trees in question are incorporated in the enumerator $T(y)$ (with additional variables if refinement of the enumeration is called for).

The object of this note is to collect a number of examples of the use of (2) which increase the store of combinatorial interpretations. Some of these are known, some are new, and many prospects are ignored since it is impossible to be exhaustive.

2. Preliminaries

A few immediate consequences of (2), which have general importance, are worth preliminary notice. First, writing

$$F_n(x) = \sum_{k=1}^{n} F_{nk} x^k$$

($F_n(x)$ is the enumerator of forests with n (labeled) points by number of trees) and $F_n(1) = F_n$, then with the usual conventions of the Blissard calculus, (2) may be rewritten

$$\exp yF(x) = \exp xT(y), \qquad F^n(x) \equiv F_n(x). \tag{3}$$

Using the exponential generating function for Bell polynomials (cf. [5], Eq. 45, p. 36), namely, in abbreviated notation

$$\exp uY(f; g) = \exp f(ug_1 + u^2 g_2/2! + \cdots)$$

with

$$Y^n(f; g) = Y_n(fg_1, fg_2, \dots, fg_n), \qquad f^k \equiv f_k,$$

it follows from (3) that

$$F_n(x) = Y_n(x; T) = Y_n(xT_1, \dots, xT_n). \tag{4}$$

Next, since $F(1, y) = \exp yF = \exp T(y)$, it follows that

$$\exp yF(x) = \exp xT(y) = (\exp yF)^x$$

and by [5], problem 24 of Chapter 2,

$$F_n(x) = Y_n[(x)F_1, \ldots, (x)F_n], \qquad (x)^k \equiv (x)_k = x(x-1)\cdots(x-k+1) \tag{5}$$

which is equation (5) of [6].

Finally, $\exp yF = \exp T(y)$ implies

$$T_n = Y_n(fF_1, \ldots, fF_n), \qquad f^k \equiv f_k = (-1)^{k-1}(k-1)! \tag{6}$$

Because of these relations, it is convenient to notice here the following congruences for the Bell polynomials appearing in L. Carlitz [1] (in different notation): with p a prime

$$Y_p(y_1, \ldots, y_p) \equiv y_1{}^p + y_p \qquad (\mathrm{mod}\, p),$$

$$Y_{n+p} \equiv Y_n Y_p + \sum_{r=1}^{n} \binom{n}{r} y_{r+p} Y_{n-r} \qquad (\mathrm{mod}\, p),$$

$$Y_{p^r} = y_1^{p^r} + y_p^{p^{r-1}} + \cdots + y_{p^r} \qquad (\mathrm{mod}\, p),$$

$$Y_{np} \equiv Y_n(Y_p, y_{2p}, \ldots, y_{np}) \qquad (\mathrm{mod}\, p),$$

$$Y_{np^r} \equiv Y_n(Y_{p^r}, y_{2p^r}, \ldots, y_{np^r}) \qquad (\mathrm{mod}\, p).$$

These are extended to the more general polynomials $Y_n(fg_1, \ldots, fg_n)$ by simple substitution, bearing in mind that f is an umbral variable; ex. gr.

$$Y_p(fg_1, \ldots, fg_p) \equiv f_p g_1{}^p + f_1 g_p \qquad (\mathrm{mod}\, p)$$

and in compressed notation

$$Y_{n+p}(f; g) \equiv Y_n(f; g)\, Y_p(f; g) + \sum_{r=1}^{n} \binom{n}{r} fg_{r+p} Y_{n-r}(f; g) \qquad (\mathrm{mod}\, p).$$

3. Forests of Unrestricted Rooted and Free Trees

It is convenient to begin with rooted trees, with all points labeled, of course, but otherwise unrestricted. Their forests have been enumerated in my paper [7] but without the detail they deserve here. The number of such trees with n points is, of course, $R_n = n^{n-1}$, $n = 1, 2, \ldots$ and

$$R(y) = \sum R_n y^n/n! = y \exp R(y)$$

by a result due to George Pólya [3]. Hence if * $A(x, y)$ is the enumerator of forests of such trees, by (2),

$$A(x, y) = \exp xR(y). \tag{7}$$

Writing $R(y) = u$, so that $ue^{-u} = y$, it is found that, using the Lagrange formula,

$$\begin{aligned} A(x, y) = \exp xu &= 1 + \sum_{n=1} \frac{y^n}{n!} \frac{d^{n-1}}{du^{n-1}} (xe^{u(n+x)})\Big|_{u=0} \\ &= 1 + \sum_{n=1} x(x+n)^{n-1} y^n/n! \end{aligned} \tag{8}$$

or $A_0(x) = 1$,

$$A_n(x) = x(x+n)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} n^{n-1-k} k x^k, \qquad n = 1, 2, \ldots \tag{8a}$$

is the enumerator of forests with n points by number of trees. Note that $A_n \equiv A_n(1) = (n+1)^{n-1}$ and also that (4) and (6) yield the identities

$$\begin{aligned} A_n(x) &= x(x+n)^{n-1} = Y_n(xR_1, \ldots, xR_n), \\ R_n &= n^{n-1} = Y_n(fA_1, \ldots, fA_n), \qquad f^k \equiv f_k = (-1)^{k-1}(k-1)! \end{aligned} \tag{9}$$

Using the suffix notation for partial derivatives, it follows from (7) that, with a prime denoting a derivative,

$$\begin{aligned} A_y(x, y) &= xR'(y)\, A(x, y), \\ A_x(x, y) &= R(y)\, A(x, y). \end{aligned} \tag{10}$$

The corresponding recurrence relations for $A_n(x)$ are

$$\begin{aligned} &A_{n+1}(x) = xR(R + A(x))^n, \quad R^n \equiv R_n, \quad A^n(x) \equiv A_n(x), \quad R_0 = 0, \\ &A'_n(x) = (R + A(x))^n. \end{aligned} \tag{11}$$

Using (8a), these become instances of two forms of Abel's generalization of the binomial formula, which I do not take space to write down. The first of equations (10) has a variation derived from

$$R'(y) = R(y)/(y - yR(y));$$

this entails

$$(y - yR(y))A_y(x, y) = xR(y)A(x, y) \tag{10a}$$

* It seems impossible to find suggestive notation adequate to the great variety of possible forests, so I start at the beginning of the alphabet.

and

$$A_n(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \left(n-k\right)^{n-k-2} (k+x)\, A_k(x), \qquad n = 1, 2, \ldots \tag{11a}$$

or

$$(x+n)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} \left(x+k\right)^{k} (n-k)^{n-k-2}, \qquad n = 1, 2, \ldots$$

a formula of the Abel type that is not so well known. It is also interesting to notice, as in [7], that

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} n^{n-1-k} k x^k$$

has the inverse

$$x^n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} k^{n-k} A_k(x). \tag{12}$$

Using $A_n(x) = Y_n(xR_1, \ldots, xR_n)$ and the Carlitz congruences given above, it turns out that

$$\begin{aligned} A_{np}(x) &\equiv x^{np} \qquad (\mathrm{mod}\ p), \\ A_{np+m}(x) &\equiv (x^p + m)^n A_m(x) \qquad (\mathrm{mod}\ p). \end{aligned} \tag{13}$$

Turn now to free trees, enumerated by $T(y) = \sum T_n y^n/n!$ with $T_0 = 1$, $T_n = n^{n-2}$, which, of course, is Cayley's formula. First notice the following formula due to Alfred Rényi [4],

$$T(y) = R(y) - R^2(y)/2,$$

with $R(y)$ the enumerator of rooted trees by number of (labeled) points. This may be derived by use of the Lagrange formula and is verified by

$$yT'(y) = yR'(y)(1 - R(y)) = R(y),$$

a formula appearing in [5, p. 139]. Then if $A^*(x, y)$ is the enumerator for forests of free trees

$$\begin{aligned} A^*(x, y) &= \exp[xR(y) - xR^2(y)/2] \\ &= \exp(xu - xu^2/2), \qquad u = R(y), \qquad ue^{-u} = y. \end{aligned} \tag{14}$$

Then first

$$\exp(xu - xu^2/2) = \exp uB(x), \qquad B^n(x) \equiv B_n(x)$$

with

$$B_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{2^n k!(n-2k)!} x^{n-k} \tag{15}$$

and by the Lagrange formula

$$f(u) = f(0) + \sum_{n=1} \frac{y^n}{n!} \frac{d^{n-1}}{du^{n-1}} [f'(u)\, e^{nu}]\Big|_{u=0}$$

and $f(u) = \exp(xu - xu^2/2) = \exp uB(x)$,

$$A^*(x, y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{n!} B(x)\big(B(x) + n\big)^{n-1}, \qquad B^k(x) \equiv B_k(x), \tag{16}$$

so that $A_0^*(x) = 1$,

$$A_n^*(x) = B(x)\big(B(x) + n\big)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} n^{n-1-k} k B_k(x), \qquad n = 1, 2, \ldots \tag{17}$$

a result equivalent to the main result of [4]. Note that (17) has the inverse

$$B_n(x) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} k^{n-k} A_k^*(x). \tag{18}$$

Note also that differentiation of $\exp uB(x) = \exp(xu - xu^2/2)$ leads to $B_0(x) = 1$,

$$B_{n+1}(x) = xB_n(x) - xnB_{n-1}(x), \qquad n = 0, 1, 2, \ldots \tag{19}$$

and that (cf. problem 17, p. 85 of [5])

$$B_n(x) = x^n C_n(1, -x^{-1}, 0, 0, \ldots)$$

with $C_n(t_1, t_2, \ldots, t_n)$ the cycle indicator of the symmetric group.

Alternatively, again with the suffix notation for partial derivatives and a prime for a derivative,

$$\begin{aligned} A_y^*(x, y) &= xR'(y)\big(1 - R(y)\big)\, A^*(x, y) = xy^{-1}R(y)\, A^*(x, y), \\ A_x^*(x, y) &= \big(R(y) - R^2(y)/2\big)\, A^*(x, y). \end{aligned} \tag{20}$$

These imply

$$\begin{aligned} nA_n^*(x) &= x\big(A^*(x) + R\big)^n, \quad [A^*(x)]^n = A_n^*(x), \quad R^n \equiv R_n (R_0 = 0), \\ [A_n^*(x)]' &= \big(A^*(x) + T\big)^n, \qquad T^n \equiv T_n = n^{n-2} (T_0 = 0). \end{aligned} \tag{21}$$

For orientation, the first few values of $A_n^*(x)$ are as follows

$$A_0^*(x) = 1, \qquad A_2^*(x) = x + x^2,$$

$$A_4^*(x) = 16x + 15x^2 + 6x^3 + x^4,$$

$$A_1^*(x) = x, \qquad A_3^*(x) = 3x + 3x^2 + x^3,$$

$$A_5^*(x) = 125x + 110x^2 + 45x^3 + 10x^4 + x^5.$$

Note that if A_{nk}^* is the coefficient of x^k in $A_n^*(x)$,

$$A_{n1}^* = T_n = n^{n-2}, \qquad A_{n2}^* = n^{n-4}(n-1)(n+6)/2,$$

$$A_{n3}^* = n^{n-6}(n-1)(n-2)(n^2+13n+60)/8$$

while

$$A_{nn}^* = 1, \qquad A_{n,n-1}^* = \binom{n}{2},$$

$$A_{n,n-2}^* = 3\binom{n+1}{4}, \qquad A_{n,n-3}^* = 15\binom{n+2}{6} + \binom{n}{4}.$$

According to [4], the limit of $A_{n,k+1}^* T_n^{-1}$ for increasing n is $2^{-k}/k!$, and hence the limit of $A_n^*(1)T_n^{-1}$ is $e^{1/2}$.

The congruences for $A_n^*(x)$ are closely similar to those for $A_n(x)$. Because $T_2 = 1 \not\equiv 0 \pmod 2$, the prime 2 needs special attention. It is found that, omitting arguments,

$$A_2^* \equiv x + x^2 \pmod 2, \qquad A_{2n}^* \equiv (x + x^2)^n \pmod 2,$$

$$A_p^* \equiv x^p \pmod p, \qquad A_{pn}^* \equiv x^{pn} \pmod p, \qquad p > 2,$$

$$A_{np+m}^* \equiv (A_p^* + m)^k A_m^* \pmod p, \qquad p > 1. \tag{22}$$

To accommodate these results to the oriented varieties it is sufficient to notice that the number of oriented rooted trees with n labeled points is $0_n = (2n)^{n-1}$ while the corresponding number of free trees is $P_n = 2^{n-1}n^{n-2}$. Thus the tree enumerators are related to the earlier ones by

$$0(y) = \tfrac{1}{2}R(2y), \qquad P(y) = \tfrac{1}{2}T(2y)$$

and the forest enumerators by

$$\begin{aligned} B(x, y) &= \exp\frac{x}{2}R(2y) = A\left(\frac{x}{2}, 2y\right), \\ B^*(x, y) &= \exp\frac{x}{2}T(2y) = A^*\left(\frac{x}{2}, 2y\right), \end{aligned} \tag{23}$$

so that

$$B_n(x) = 2^n A_n\left(\frac{x}{2}\right),$$
$$B_n^*(x) = 2^n A_n^*\left(\frac{x}{2}\right). \tag{24}$$

4. Trees of Least Height

A second category of enumeration is "least height" for rooted trees and "least diameter" for free trees. For rooted trees, the least height is zero for one point, and one otherwise; the number of such trees with n labeled points is $u_n = n$, and the enumerator is $u(y) = ye^y$. For free trees, the least diameter is zero for one point, one for two points, and two otherwise, and if v_n is the number with n labeled points, $v_1 = v_2 = 1$, $v_n = n$, $n = 3, 4, \ldots$ so that $v(y) = ye^y - y^2/2$.

Then, if $L(x, y)$ is the enumerator of forests of rooted trees of least height, $L^*(x, y)$ is the enumerator of forests of free trees of least diameter,

$$L(x, y) = \exp(xye^y),$$
$$L^*(x, y) = \exp(xye^y - xy^2/2). \tag{25}$$

These give at once

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} k^{n-k} x^k,$$
$$L_n^*(x) = \sum_{k=0}^{N} \binom{n}{2k} \frac{(2k)!}{2^k k!} (-x)^k L_{n-2k}(x), \qquad N = [n/2]. \tag{26}$$

Note that the inverse of the first of these is

$$x^n = \sum_{k=0}^{n} (-1)^{n+k} A_{nk} L_k(x),$$

with $A_{nk} = \binom{n}{k} n^{n-1-k} k$, the number of forests of unrestricted rooted trees with n labeled points and k trees, as given above. Also, if $L_n(x) = \sum L_{nk} x^k$, equation (12) may now be rewritten

$$x^n = \sum_{k=0}^{n} (-1)^{n+k} L_{nk} A_k(x).$$

Denoting partial derivatives by suffixes, it follows from equation (25) that

$$L_y(x, y) = x(1 + y)e^y L(x, y), \qquad L_x(x, y) = ye^y L(x, y),$$
$$L_y^*(x, y) = x[(1 + y)e^y - y]\, L^*(x, y), \quad L_x^*(x, y) = (ye^y - y^2/2)\, L^*(x, y). \tag{27}$$

These imply (a prime denotes a derivative) the recurrences

$$L_{n+1}(x) = x[(L + 1)^n + n(L + 1)^{n-1}], \qquad L^n \equiv L_n(x),$$
$$L_n'(x) = n(L + 1)^{n-1},$$
$$L_{n+1}^*(x) = x[(L^* + 1)^n + n(L^* + 1)^{n-1} - nL_{n-1}^*(x)], \qquad (L^*)^n = L_n^*(x), \tag{28}$$
$$(L_n^*)' = n(L^* + 1)^{n-1} - \binom{n}{2} L_{n-2}^*(x).$$

The first of (28) in ordinary notation reads, after a little simplification,

$$L_{n+1}(x) = x \sum_{k=0}^{n} (k + 1)\binom{n}{k} L_{n-k}(x), \tag{29}$$

while the third is

$$L_{n+1}^*(x) = x \sum_{k=0}^{n} (k + 1)\binom{n}{k} L_{n-k}^*(x) - xnL_{n-1}^*(x). \tag{30}$$

Since $L_{nj} = \binom{n}{j} j^{n-j}$, (29) implies the identity

$$\binom{n+1}{j+1}(j + 1)^{n-j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{j}(k + 1)\, j^{n-k-j}.$$

There seems to be no simple formula for the L_{nk}^* but it is worth noting the special results

$$L_{n1}^* = n, \qquad n > 2,$$
$$L_{n2}^* = \binom{n}{2} 2^{n-2} - 3\binom{n}{3}, \qquad n > 4,$$
$$L_{n3}^* = \binom{n}{3} 3^{n-3} - 6\binom{n}{4} 2^{n-4} + 15\binom{n}{5}, \qquad n > 5,$$
$$L_{nn}^* = 1, \qquad L_{n,n-2}^* = 3\binom{n+1}{4},$$
$$L_{n,n-1}^* = \binom{n}{2}, \qquad L_{n,n-3}^* = 15\binom{n+2}{6} - 11\binom{n}{4}.$$

The congruences are

$$
\begin{aligned}
&L_{np}(x) \equiv x^{np} \qquad (\text{mod } p),\\
&L_{n+p}(x) \equiv L_n(x)\, L_p(x) + xL_n'(x) \qquad (\text{mod } p),\\
&L_{np}^*(x) \equiv \big(L_p^*(x)\big)^n \qquad (\text{mod } p),\\
&L_{n+p}^*(x) \equiv L_n^*(x)\, L_p^*(x) + x\big(L_n^*(x)\big)' + x\binom{n}{2} L_{n-2}^*(x) \qquad (\text{mod } p),
\end{aligned}
\tag{31}
$$

with

$$L_2^*(x) \equiv x + x^2 \qquad (\text{mod } 2), \qquad L_p^*(x) \equiv x^p \qquad (\text{mod } p), \qquad p > 2.$$

The oriented variety enumerators are found exactly as before.

5. Trees of Greatest Height

The greatest height of a rooted tree with p points is $p - 1$, and the number of such labeled trees (cf. [8]) is $p!$, $p = 1, 2, \ldots$. The greatest spread of a (free) tree with p points is also $p - 1$, and the number of such labeled trees is $p!/2$, $p = 2, 3, \ldots$ while for $p = 1$ the number is 1. The corresponding enumerators are

$$y(1 - y)^{-1} \qquad \text{and} \qquad y + y^2(1 - y)^{-1}2^{-1}.$$

Taking the forest enumerators as $G(x, y)$, $G^*(x, y)$, then

$$
\begin{aligned}
&G(x, y) = \exp xy(1 - y)^{-1},\\
&G^*(x, y) = \exp\big(xy + xy^2(1 - y)^{-1}2^{-1}\big) = \exp(xy/2)\, G\left(\frac{x}{2}, y\right).
\end{aligned}
\tag{32}
$$

From the first of these it follows at once that $G_0(x) = 1$ and

$$G_n(x) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1} x^k, \qquad n = 1, 2, \ldots \tag{33}$$

The coefficient of x^k in $G_n(x)$ is the signless Lah number, $(-1)^n L_{nk}$ in the notation of [5, p. 43]. Also $G_n(x) = n! L_n^{(-1)}(-x)$ with $L_n^p(x)$ the generalized Laguerre polynomial.

Hence asymptotically*

$$G_n \sim n^{n-\frac{1}{4}} e^{-n+2\sqrt{n}}/\sqrt{2e}. \tag{34}$$

Also the Euler transform of $G(x, y)$ is

$$\begin{aligned} H(x, y) &= (1 + y)^{-1} G(x, y(1 + y)^{-1}) = (1 + y)^{-1} \exp xy \\ &= \exp y(x - D - 1), \qquad D^n \equiv D_n = D^n 0! \end{aligned} \tag{35}$$

with D_n a rencontre (subfactorial) number. This implies the inverse relations

$$\begin{aligned} H_n(x) + nH_{n-1}(x) &= x^n, \\ H_n(x) = (x - D - 1)^n &= \sum_{k=0}^{n} (-1)^k (n)_k \, x^{n-k}, \end{aligned} \tag{36}$$

with $(n)_k = n(n-1) \cdots (n-k+1)$, and also

$$\begin{aligned} G_n(x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} H_k(x), \\ H_n(x) &= \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \frac{n!}{k!} G_k(x). \end{aligned} \tag{37}$$

Since $H_k(1) = (-1)^n D_n$, the first of (37) gives

$$G_n = G_n(1) = n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{D_k}{k!}.$$

Recurrences for the $G_n(x)$ and $G_n^*(x)$ follow from

$$\begin{aligned} G_y(x, y) &= x(1 - y)^{-2} G(x, y), \\ G_y^*(x, y) &= x(1 - y + y^2/2)(1 - y)^{-2} G^*(x, y), \end{aligned} \tag{38}$$

and read

$$\begin{aligned} &G_{n+1}(x) - (x + 2n) G_n(x) + (n)_2 G_{n-1}(x) = 0, \\ &G_{n+1}^*(x) - (x + 2n) G_n^*(x) + n(x + n - 1) G_{n-1}^*(x) - x \binom{n}{2} G_{n-2}^*(x) = 0, \end{aligned} \tag{39}$$

* I owe this remark to my colleagues J. H. Van Lint and L. A. Shepp.

so that

$$G_{n+1} = G_{n+1}(1) = (2n+1)\,G_n - (n)_2\,G_{n-1}\,,$$
$$G^*_{n+1} = G^*_{n+1}(1) = (2n+1)\,G^*_n - n^2 G^*_{n-1} + \binom{n}{2} G^*_{n-2}\,. \tag{40}$$

By the second of (32)

$$G^*_n(2x) = \big(G(x) + x\big)^n = \sum_{k=0}^{n} \binom{n}{k} G_{n-k}(x)\,x^k, \tag{41}$$

which implies

$$G^*_{nk} = \frac{n!}{2^k k!} \sum_{j=0}^{K} \binom{k}{j}\binom{n-j-1}{k-j-1}, \tag{42}$$

which is equivalent to

$$G^*_{nk} = \frac{n!}{2^k k!} \sum_{j=0}^{K} (-1)^j \binom{k}{j}\binom{n+k-1-2j}{k-1-2j}, \qquad K = [\tfrac{1}{2}(k-1)].$$

The congruences for the polynomials are

$$G_{np}(x) \equiv x^{np} \pmod{p}, \qquad G_{np+m}(x) \equiv x^{np} G_m(x) \pmod{p},$$
$$G^*_2(x) \equiv x + x^2 \pmod{2}, \qquad G^*_{2n}(x) \equiv \big(G^*_2(x)\big)^n \pmod{2},$$
$$G^*_{2n+1}(x) \equiv (G^*_2)^{2m}(G^*_2 + 1)\,G^*_1 \pmod{2}, \qquad n = 2m+1 \tag{43}$$
$$\equiv (G^*_2)^{2m}\,G^*_1 \pmod{2}, \qquad n = 2m,$$
$$G^*_{np}(x) \equiv x^{np} \pmod{p}, \qquad G^*_{np+m} \equiv x^{np} G^*_m(x) \pmod{p}, \qquad p > 2.$$

6. Rooted Trees of Limited Height

If the enumerator of labeled rooted trees of height at most h is taken as $S(y; h)$, then by equation (20a) of [8]

$$S(y; h) = y \exp S(y; h-1), \tag{44}$$

with $S(y; 0) = 1$, $S(y; 1) = ye^y = u(y)$ (in the notation of Section 4 above). The enumerator of forests is then given by

$$H(x, y; h) = \exp xS(y; h) = \exp xy \exp S(y; h-1)$$
$$= \exp xyH(1, y; h-1). \tag{45}$$

Hence

$$H_y(x, y; h) = x[H(1, y; h-1) + yH_y(1, y; h-1)]H(x, y; h) \quad (46)$$

and

$$H_{n+1}(x; h) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(x; h)(k+1)\, H_k(1; k-1). \quad (47)$$

Thus the enumerators $H_n(x; h)$ of forests with n labeled points of rooted trees of height at most h are fully determined when the numbers $H_k(1; h-1)$, $h = 0(1)n - 1$ are known. But by (44) and (45)

$$S(y; h) = yH(1; y, h-1), \quad (48)$$

which implies

$$S_{nh} = nH_{n-1}(1; h-1). \quad (49)$$

The numbers on the right are the cumulations of the numbers in Table 3 of [8], which have been used to obtain the numbers in Table 1 below; this gives $H_n(1; h)$ for $h = 1(1)n - 1$, $n = 0(1)8$. Of course for $h > n - 1$,

$$H_n(1; h) = H_n(1; n-1) = (n+1)^{n-1}.$$

TABLE 1

THE NUMBER, $H_n(1; h)$, OF FORESTS OF LABELED ROOTED TREES OF HEIGHT AT MOST h, WITH n POINTS

h/n	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1			3	10	41	196	1057	6322	41393
2				16	101	756	6607	65794	733833
3					125	1176	12847	160504	2261289
4						1296	16087	229384	3687609
5							16807	257104	4480569
6								262144	4742649
7									4782969

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